TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 293, Number 2, February 1986

A SOLUTION OF WARNER'S 3RD PROBLEM FOR REPRESENTATIONS OF HOLOMORPHIC TYPE[1]

BY
FLOYD L. WILLIAMS

ABSTRACT. In response to one of ten problems posed by G. Warner, we assign (to the extent that it is possible) a geometric or cohomological interpretation—in the sense of Langlands—to the multiplicty in $L^2(\Gamma\backslash G)$ of an irreducible unitary representation π of a semisimple Lie group G, where Γ is a discrete subgroup of G, in the case when π has a highest weight.

1. Introduction. Given a discrete subgroup Γ of a semisimple Lie group G and a special class of nontempered unitary representations π of G, we express the multiplicity of π in $L^2(\Gamma\backslash G)$ as the dimension of the sheaf cohomology of a line bundle over $\Gamma\backslash G$ modulo a compact Cartan subgroup of G, and we find the best vanishing theorem for this cohomology. These prototype results extend those known in the case when π is a discrete series representation.

In more detail let G be connected noncompact and linear, and assume Γ is cocompact in G and torsion free. We assume G has the rank of a maximal compact subgroup K and we fix a Cartan subgroup H of G contained in K. Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}$ denote the complexified Lie algebras of G, K, H respectively, let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the set of nonzero roots of $\mathfrak{g}$ relative to $\mathfrak{h}$, let $\mathfrak{g}_\beta$ be the root space of $\beta \in \Delta$, and let Δ_k, Δ_n be the set of compact, noncompact roots respectively in Δ. Given a choice of a positive system of roots Δ^+ in Δ one can assign a unique G-invariant holomorphic structure on the quotient $D = G/H$ such that the space of antiholomorphic tangent vectors at the origin is given by

$$\mathfrak{n} = \sum_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}. \tag{1.1}$$

A homogeneous line bundle $\mathcal{L}_\lambda \to D$ over D induced by a character λ of H also has a G-invariant holomorphic structure and thus we can consider for $Y = \Gamma\backslash D$ the space $H^q(Y, L_\lambda)$ of Γ-invariant $\mathcal{L}_\lambda$-valued harmonic C^∞ forms of type $(0, q)$ on D, relative to some G-invariant Hermitian metrics on $D, \mathcal{L}_\lambda$. Alternately $H^q(Y, \mathcal{L}_\lambda)$ is the qth-dimensional cohomology of Y with coefficients in the sheaf $\mathcal{O}_\lambda$ of local holomorphic sections of $\Gamma\backslash\mathcal{L}_\lambda$. Define

$$q(\lambda) = |\{\alpha \in \Delta_k^+ : (\lambda+\delta, \alpha) < 0\}| + |\{\alpha \in \Delta_n^+ : (\lambda+\delta, \alpha) > 0\}| \tag{1.2}$$

where $\Delta_k^+ = \Delta^+ \cap \Delta_k$, $\Delta_n^+ = \Delta^+ \cap \Delta_n$, $2\delta = \langle\Delta^+\rangle$ (the sum of roots in Δ^+), $(\cdot)$ is the Killing form of $\mathfrak{g}$, and where $|S|$ denotes the cardinality of a set S. Assume that $\lambda + \delta$ is regular and let $\pi_{\lambda+\delta} \in \hat{G}$ (the unitary dual of G) be the corresponding

Received by the editors February 5, 1985. A portion of these results were included in a presentation to the A.M.S. 822nd meeting, October 26-27, 1985.
1980 *Mathematics Subject Classification*. Primary 22E45; Secondary 32L10, 32M15.
[1] Research supported by NSF Grant No. PRM 8205819.

Harish-Chandra discrete series representation of G. If $m_{\lambda+\delta}(\Gamma)$ denotes the multiplicity of $\pi_{\lambda+\delta}$ in $L^2(\Gamma\backslash G)$ then for a very mild condition on λ (see (2.10) below) we know that

$$\text{(1.3)}\qquad \begin{aligned} &\text{(i) } H^q(Y, \mathcal{L}_\lambda) = 0 \quad \text{for } q \neq q(\lambda) \text{ in (1.2)},\\ &\text{(ii) } \dim H^{q(\lambda)}(Y, \mathcal{L}_\lambda) = m_{\lambda+\delta}(\Gamma). \end{aligned}$$

(1.3) is the solution of a conjecture of Langlands [**2, 3**]—obtained in part in [**5**] and fully in [**9**]. It is natural to consider therefore the extent to which analogues of (1.3) are possible for a nondiscrete series $\pi \in \hat{G}$. This is the content of problem 3 of Warner [**6**, p. 139]. Given such a π one of the first steps is to find a candidate $q(\lambda, \pi)$ for the role of $q(\lambda)$.

In this paper we assume that G/K has a G-invariant holomorphic structure compatible with the choice Δ^+ above and we assume that $\pi \in \hat{G}$ is a highest weight representation with a nonsingular infinitesimal character. Then we find the right candidate $q(\lambda, \pi)$ and the analogue of (1.3)(ii). Regarding (1.3)(i) our results indicate however that one must expect, in general, the existence of nonzero cohomology in dimensions $\neq q(\lambda, \pi)$ (which prevents the immediate computation of $m_\pi(\Gamma)$ by an index formula). We determine all such dimensions. The main result is Theorem 3.9, which appears in sharper form in Theorem 3.13. Theorem 3.9 relies basically on results developed in [**4, 7, 8**].

2. More notation. As before we shall write $\langle Q\rangle$ for the sum of roots in a subset Q of Δ. We denote by W, W_k the Weyl groups of $(\mathfrak{g}, \mathfrak{h}), (\mathfrak{k}, \mathfrak{h})$ respectively, and by $\mathcal{F}$ the set of integral linear forms on the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$. Let $\Delta^+ \subset \Delta$ be as before, a positive system which we fix from now on. For $\Lambda \in \mathfrak{h}^*$ such that $\Lambda + \delta$ is regular, we set

$$\text{(2.1)}\qquad \begin{aligned} &P^{(\Lambda)} = \{\alpha \in \Delta: (\Lambda + \delta, \alpha) > 0\}, \qquad 2\delta^{(\Lambda)} = \langle P^{(\Lambda)}\rangle,\\ &Q_\Lambda = \{\alpha \in \Delta_n^+: (\Lambda + \delta, \alpha) > 0\}, \qquad Q'_\Lambda = \Delta_n^+ - Q_\Lambda,\\ &\mathfrak{b}_\Lambda = \mathfrak{h} + \sum_{\alpha \in P^{(\Lambda)}} \mathfrak{g}_\alpha \quad \text{(a Borel subalgebra)}. \end{aligned}$$

Define

$$\text{(2.2)}\qquad \begin{aligned} \mathcal{F}'_0 = \mathcal{F}'_0(\Delta^+) = \{\Lambda \in \mathcal{F}: \Lambda + \delta \text{ is regular and } \Delta_k^+\text{-dominant}\};\\ 2\delta_k = \langle \Delta_k^+\rangle, \quad \delta_n = \delta - \delta_k. \end{aligned}$$

If $\theta = m + u$ is a Levi decomposition of a parabolic subalgebra θ of $\mathfrak{g}$ with unipotent radical u and a reductive complement m, we let $\Delta(m), \Delta(u)$ denote the roots of m, u respectively and we set

$$\text{(2.3)}\qquad \theta_{u,n} = \text{the set of noncompact roots in } \Delta(u).$$

Thus if $P \subset \Delta$ is any positive system and θ contains the Borel subalgebra $\mathfrak{h} + \sum_{\alpha \in P} \mathfrak{g}_\alpha$ then

$$\text{(2.4)}\qquad m = \mathfrak{h} + \sum_{\alpha \in \Delta(m)} \mathfrak{g}_\alpha, \quad u = \sum_{\alpha \in P - \Delta(m)} \mathfrak{g}_\alpha, \quad \theta_{u,n} = P_n - \Delta(m)$$

where $P_n = P \cap \Delta_n$. In particular if $\theta \supset \mathfrak{b}_\Lambda$ for $\Lambda \in \mathcal{F}'_0$ we define

$$\text{(2.5)}\qquad q(\Lambda; \theta) = 2|\theta_{u,n} \cap Q_\Lambda| - |\theta_{u,n}| + |Q'_\Lambda|.$$

Then $q(\Lambda;\theta)$ is a nonnegative integer. A collection of parabolic subalgebras $\{\theta_i\}_{i=1}^t$ each containing $\mathfrak{b}_\Lambda$, $\Lambda \in \mathcal{F}_0'$, is called *representative* if $\theta_{u_i,n} \neq \theta_{u_j,n}$ for $i \neq j$, $1 \leq i, j \leq t$, and if for any parabolic subalgebra $\theta \supset \mathfrak{b}_\Lambda$ we have $\theta_{u,n} = \theta_{u_i,n}$ for some i. This definition is equivalent to that given in [**8**]; we thank Professor Kostant for pointing this out to us. Given such a collection we have the following important sets: For an integer $q \geq 0$

$$S(\Lambda, q) \overset{\text{def}}{=} \{i : q(\Lambda;\theta_i) = q \text{ and } \theta_{u_i,n} = \theta_{u,n} \text{ for some parabolic subalgebra } \theta = m + u \supset \mathfrak{b}_\Lambda \text{ with } (\Lambda + \delta - \delta^{(\Lambda)}, \Delta(m)) = 0\}. \tag{2.6}$$

PROPOSITION 2.7. *Let $\{\theta_i\}_{i=1}^t$ be a representative set of parabolics $\supset \mathfrak{b}_\Lambda$ for $\Lambda \in \mathcal{F}_0'$, and let $\theta = m + u \supset \mathfrak{b}_\Lambda$ be any parabolic such that $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(m)) = 0$. Let i_0 be the unique index, $1 \leq i_0 \leq t$, such that $\theta_{u,n} = \theta_{u_{i_0},n}$. Then $i_0 \in S(\Lambda, q(\Lambda;\theta))$. If moreover Λ satisfies the condition*

$$(\Lambda + \delta - \delta^{(\Lambda)}, \alpha) \neq 0 \quad \text{for } \alpha \in Q_\Lambda \cup \theta_{u,n} \tag{2.8}$$

then in fact $Q_\Lambda \subset \theta_{u,n}$, $q(\Lambda;\theta) = |Q_\Lambda| + |\Delta_n^+| - |\theta_{u,n}|$ and $S(\Lambda, q(\Lambda;\theta)) = \{i_0\}$.

Note: Since $\Lambda + \delta - \delta^{(\Lambda)}$ is $P^{(\Lambda)}$-dominant and $Q_\Lambda \cup \theta_{u,n} \subset P_n^{(\Lambda)} \subset P^{(\Lambda)}$ (2.8) is equivalent to

$$(\Lambda + \delta - \delta^{(\Lambda)}, \alpha) > 0 \quad \text{for } \alpha \in Q_\Lambda \cup \theta_{u,n}. \tag{2.9}$$

PROOF OF PROPOSITION 2.7. By the definitions involved $i_0 \in S(\Lambda, q(\Lambda;\theta))$. Conversely let $i \in S(\Lambda, q(\Lambda;\theta))$ so $q(\Lambda;\theta_i) \overset{\text{(i)}}{=} q(\Lambda;\theta)$ and $\theta_{u_i,n} \overset{\text{(ii)}}{=} \theta'_{u',n}$ for some $\theta' = u' + m' \supset \mathfrak{b}_\Lambda$ for which $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(m')) \overset{\text{(iii)}}{=} 0$. Let $N_\Lambda = \{\alpha \in P_n^{(\Lambda)} : (\Lambda + \delta - \delta^{(\Lambda)}, \alpha) = 0\}$. Since $\Lambda + \delta - \delta^{(\Lambda)}$ is $P^{(\Lambda)}$-dominant (as just noted) and since $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(m)) = 0$,

$$\begin{aligned} P_n^{(\Lambda)} &= N_\Lambda \cup (P_n^{(\Lambda)} - \Delta(m)) \overset{\text{(iv)}}{=} N_\Lambda \cup \theta_{u,n} \quad \text{(cf. (2.4))} \\ &\Rightarrow Q_\Lambda = Q_\Lambda \cap N_\Lambda \cup Q_\Lambda \cap \theta_{u,n}. \end{aligned}$$

By (2.8)

$$N_\Lambda \cap \theta_{u,n} = Q_\Lambda \cap N_\Lambda = \varnothing \Rightarrow n \overset{\text{def}}{=} |\Delta_n^+| = |P_n^{(\Lambda)}| \overset{\text{(v)}}{=} |N_\Lambda| + |\theta_{u,n}|$$

and $Q_\Lambda \subset \theta_{u,n}$. Then $q(\Lambda;\theta) = |Q_\Lambda| + n - |\theta_{u,n}|$ by (2.5). Similarly from (iii)

$$\begin{aligned} P_n^{(\Lambda)} &\overset{\text{(vi)}}{=} (N_\Lambda - \theta'_{u',n}) \cup \theta'_{u',n} \Rightarrow Q_\Lambda = Q_\Lambda \cap \theta'_{u',n} \quad (\text{again since } Q_\Lambda \cap N_\Lambda = \varnothing) \\ &\Rightarrow q(\Lambda;\theta') = |Q_\Lambda| + n - |\theta'_{u',n}|, \end{aligned}$$

so by (i), (ii)

$$q(\Lambda;\theta) = q(\Lambda;\theta') \Rightarrow |\theta_{u,n}| \overset{\text{(vii)}}{=} |\theta'_{u',n}|.$$

Then from (v), (vi),

$$\begin{aligned} |N_\Lambda| + |\theta_{u,n}| = |N_\Lambda - \theta'_{u',n}| + |\theta'_{u',n}| &\Rightarrow |N_\Lambda| = |N_\Lambda - \theta'_{u',n}| \\ &\Rightarrow N_\Lambda = N_\Lambda - \theta'_{u',n} \Rightarrow N_\Lambda \cap \theta'_{u',n} = \emptyset, \end{aligned}$$

so by (iv) $\theta'_{u',n} \subset \theta_{u,n}$; i.e. by (ii), (vii) $\theta_{u_i,n} = \theta_{u,n}$. But $\theta_{u,n} = \theta_{u_{i_0},n}$; hence $i = i_0$, as desired. Q.E.D.

For the sake of completeness we give the "mild condition" on λ referred to prior to (1.3). It is (cf. (2.9))

$$(2.10) \qquad (\lambda + \delta - \delta^{(\lambda)}, \alpha) > 0 \quad \text{for each } \alpha \in P_n^{(\lambda)}.$$

3. A vanishing theorem and statement of the main result. We assume now that G/K has a G-invariant holomorphic structure so fixed that the space of antiholomorphic tangent vectors at the origin is given by

$$(3.1) \qquad \mathfrak{p}^- = \sum_{\alpha \in \Delta_n^+} \mathfrak{g}_{-\alpha}$$

(cf. (1.1)). This is equivalent to requiring that every $\alpha \in \Delta_n^+$ should be *totally positive*; i.e. $\alpha + \beta \in \Delta$ for $\beta \in \Delta_k \Rightarrow \alpha + \beta \in \Delta_n^+$. Then also

$$(3.2) \qquad G/H \to G/K \text{ and } \Gamma\backslash G/H \to \Gamma\backslash G/K \text{ are holomorphic fibrations.}$$

As we shall wish to make immediate applications of the results of [**7**, **8**] we assume also that the complexification of G is simply connected. In general if $P = P_k \cup P_n \subset \Delta$ is a system of positive roots compatible with a G-invariant holomorphic structure on G/K (i.e. every $\alpha \in P_n$ is totally positive), so is the conjugate system $P_k \cup -P_n$ which we shall denote by $\overline{P}$. If $\lambda \in \mathcal{F}$ we also denote by λ the character of H whose differential is λ and thus we consider the cohomology $H^q(Y, L_\lambda)$ defined in §1. As $\Lambda \in \mathcal{F}_0'$ is the Δ_k^+-highest weight of an irreducible K module the sheaf $\mathcal{O}_\Lambda \to \Gamma\backslash G/K$ and cohomology $H^q(\Gamma\backslash G/K, \mathcal{O}_\Lambda)$ are defined similarly.

THEOREM 3.3. *Let $\lambda \in \mathfrak{h}^*$ be an integral form. If $(\lambda + \delta_k, \alpha) = 0$ for some $\alpha \in \Delta_k^+$ then $H^q(Y, \mathcal{L}_\lambda) = 0$ for every q. Assume $\lambda + \delta$ is regular (so in particular $(\lambda + \delta_k, \alpha) \neq 0$ for every $\alpha \in \Delta_k^+$) and assume every $\alpha \in P_n^{(\lambda)}$ is totally positive (cf. (2.1)). Then $H^q(Y, L_\lambda) = 0$ unless q has the specific form*

$$(3.4) \qquad \begin{aligned} q(\lambda, \theta) &\overset{\text{def}}{=} |\{\alpha \in \Delta_k^+ : (\lambda + \delta, \alpha) < 0\}| + q(\lambda; \theta) \\ &= |\{\alpha \in \Delta_k^+ : (\lambda + \delta, \alpha) < 0\}| + 2|\theta_{u,n} \cap Q_\lambda| - |\theta_{u,n}| + |Q'_\lambda| \end{aligned}$$

for some parabolic subalgebra $\theta = m + u \supset \mathfrak{b}_\lambda$ for which $(\lambda + \delta - \delta^{(\lambda)}, \Delta(m)) = 0$; see (2.1), (2.3), (2.4), (2.5). On the other hand suppose we are given a parabolic subalgebra $\theta = m + u \supset \mathfrak{b}_\lambda$ for which $(\lambda + \delta - \delta^{(\lambda)}, \Delta(m)) = 0$ (where we still assume $\lambda + \delta$ is regular and every $\alpha \in P_n^{(\lambda)}$ is totally positive). Let $\{\theta_i = m_i + u_i\}_{i=1}^t$ be a representative collection of parabolic subalgebras containing $\mathfrak{b}_\lambda$ (as defined earlier) and define for an integer $q \geq 0$

$$(3.5) \qquad \begin{aligned} S(\lambda, q) = \{i : q(\lambda; \theta_i) = q \text{ and } \theta_{u_i,n} = \theta'_{u',n} \\ \text{for some parabolic subalgebra } \theta' = m' + u' \supset \mathfrak{b}_\lambda \\ \text{with } (\lambda + \delta - \delta^{(\lambda)}, \Delta(m')) = 0\}. \end{aligned}$$

Let $\mu(\lambda, \theta_i)$ be the unique Δ_k^+-dominant integral element in the W_k-orbit of $\lambda + \delta - \delta^{(\lambda)} + \langle \theta_{u_i,n} \rangle$. Then for $i \in S(\lambda, q)$ the irreducible G-representation $\pi_{\mu(\lambda,\theta_i)}$ with $\Delta_k^+ \cup -P_n^{(\lambda)}$-highest weight $\mu(\lambda, \theta_i)$ is unitarizable and the multiplicities $m_{\pi_{\mu(\lambda,\theta_i)}}$ in $L^2(\Gamma\backslash G)$ satisfy

$$(3.6) \qquad \dim H^{q(\lambda,\theta)}(Y, \mathcal{L}_\lambda) = \sum_{i \in S(\lambda, q(\lambda;\theta))} m_{\pi_{\mu(\lambda,\theta_i)}}(\Gamma).$$

PROOF. The fibration $\Gamma\backslash G/H \to \Gamma\backslash G/K$ in (3.2) gives rise to a spectral sequence which degenerates, by the Borel-Weil Theorem. If $\lambda + \delta$ is regular and $\Lambda(\lambda) \overset{\text{def}}{=} w(\lambda + \delta_k) - \delta_k \in \mathcal{F}_0'$, where $w \in W_k$ is the unique element such that $(w(\lambda + \delta_k), \Delta_k^+) > 0$, then one obtains, consequently,

$$H^q(Y, \mathcal{L}_\lambda) = H^{q-q_0}(\Gamma\backslash G/K, \mathcal{O}_{\Lambda(\lambda)}) \quad \text{for } q \geq 0 \tag{3.7}$$

where $q_0 = |w(-\Delta_k^+) \cap \Delta_k^+| = |\{\alpha \in \Delta_k^+ : (\lambda + \delta, \alpha) < 0\}|$. Moreover $H^q(Y, \mathcal{L}_\lambda) = 0$ if $\lambda + \delta_k$ is Δ_k-singular; cf. Corollaries 5.15, 5.16 and Theorem 5.24 of [**1**]. Note that by our assumptions on Δ^+, $(\delta_n, \Delta_k^+) = 0 \Rightarrow q_0 = |\{\alpha \in \Delta_k^+ : (\lambda + \delta_k, \alpha) < 0\}|$, also. From the main result of [**7**, Theorem 2.3] applied to the right-hand side in (3.7), one obtains the vanishing results stated in Theorem 3.3. Consider the second part of Theorem 3.3. $\{w\theta_i\}_{i=1}^t$ is a representative collection of parabolic subalgebras containing $\mathfrak{b}_{\Lambda(\lambda)}$, where $P^{(\Lambda(\lambda))} \overset{\text{a.}}{=} wP^{(\lambda)}$, $(w\theta_i)_{wu_i,n} \overset{\text{(i)}}{=} w\theta_{u_i,n}$ (since $w \in W_k \Rightarrow wP_n^{(\lambda)} = P_n^{(\lambda)}$, as every $\alpha \in P_n^{(\lambda)}$ is totally positive), and $Q_{\Lambda(\lambda)} = wQ_\lambda$; hence $q(\lambda; \theta_i) \overset{\text{(ii)}}{=} q(\Lambda(\lambda); w\theta_i)$; see (2.1), (2.3), (2.5). If we put

$$\begin{aligned}\underline{S}(\Lambda(\lambda), q) = \{i : q(\Lambda(\lambda); w\theta_i) = q \text{ and } (w\theta_i)_{wu_i,n} = \mathfrak{p}_{l,v} \\ \text{for some parabolic subalgebras } \mathfrak{p} = l + v \supset \mathfrak{b}_{\Lambda(\lambda)} \\ \text{with } (\Lambda(\lambda) + \delta - \delta^{(\Lambda(\lambda))}, \Delta(l)) = 0\},\end{aligned}$$

where $q \geq 0$ is an integer, then by Theorem 3.26 of [**8**] the irreducible G-representation π_{μ_i} with $\overline{P}(\Lambda(\lambda))$-highest weight

$$\mu_i \overset{\text{def}}{=} \Lambda(\lambda) + \delta - \delta^{(\Lambda(\lambda))} + \langle (w\theta_i)_{wu_i,n} \rangle$$

is unitarizable for $i \in \underline{S}(\Lambda(\lambda), q)$ and the multiplicities $m_{\pi_{\mu_i}}(\Gamma)$ in $L^2(\Gamma\backslash G)$ satisfy

$$\dim H^q(\Gamma\backslash G/K, \mathcal{O}_{\Lambda(\lambda)}) = \sum_{i \in \underline{S}(\Lambda(\lambda), q)} m_{\pi_{\mu_i}}(\Gamma) \tag{3.8}$$

for $q \geq 0$. Now (ii) and the fact that $\Lambda(\lambda) + \delta - \delta^{(\Lambda(\lambda))} \overset{\text{(iii)}}{=} w(\lambda + \delta - \delta^{(\lambda)})$ imply that $S(\lambda, q) = \underline{S}(\Lambda(\lambda), q)$. Also

$$\overline{P}^{(\Lambda(\lambda))} \overset{\text{def}}{=} \Delta_k^+ \cup -P_n^{(\Lambda(\lambda))} = \Delta_k^+ \cup -wP_n^{(\lambda)} \overset{\text{(iv)}}{=} \Delta_k^+ \cup -P_n^{(\lambda)},$$

and by (i), (iii) $\mu_i = w(\lambda + \delta - \delta^{(\lambda)} + \langle \theta_{u_i,n} \rangle)$; i.e. μ_i is the unique Δ_k^+-dominant integral element in the W_k-orbit of $\lambda + \delta - \delta^{(\lambda)} + \langle \theta_{u_i,n} \rangle$ so by definition of $\mu(\lambda, \theta_i)$, $\mu(\lambda, \theta_i) = \mu_i$. Note that (iv) can also be written as $\Delta_k^+ \cup -P_n^{(\lambda)} = w\overline{P}^{(\lambda)}$. Theorem 3.3 now follows (i.e. (3.6) follows) by (3.7), (3.8) if we take q there equal to $q(\lambda, \theta), q(\lambda; \theta)$ respectively, since $q(\lambda, \theta) - q_0 = q(\lambda; \theta)$ by (3.4). Q.E.D.

A corollary of Theorem 3.3 is the following main result.

THEOREM 3.9. *Let $\lambda \in \mathfrak{h}^*$ be an integral form such that $\lambda + \delta$ is regular and such that every $\alpha \in P_n^{(\lambda)}$ is totally positive (cf. (2.1)). Let $\theta = m + u$ be a parabolic subalgebra containing the Borel subalgebra $\mathfrak{b}_\lambda = \mathfrak{h} + \sum_{\alpha \in P^{(\lambda)}} \mathfrak{g}_\alpha$ for which $(\lambda + \delta - \delta^{(\lambda)}, \Delta(m)) = 0$ and let $\mu(\lambda, \theta)$ be the unique Δ_k^+-dominant integral element in the W_k-orbit of $\lambda + \delta - \delta^{(\lambda)} + \langle \theta_{u,n} \rangle$; cf. (2.1), (2.3), (2.4). Then the irreducible*

G-representation $\pi_{\mu(\lambda,\theta)}$ with $\Delta_k^+ \cup -P_n^{(\lambda)}$-highest weight $\mu(\lambda,\theta)$ is unitarizable and if, in particular, λ satisfies

$$(\lambda + \delta - \delta^{(\lambda)}, \alpha) \neq 0 \quad \text{for } \alpha \in Q_\lambda \cup \theta_{u,n} \tag{3.10}$$

then the multiplicity $m_{\pi_{\mu(\lambda,\theta)}}(\Gamma)$ of $\pi_{\mu(\lambda,\theta)}$ in $L^2(\Gamma\backslash G)$ satisfies

$$m_{\pi_{\mu(\lambda,\theta)}}(\Gamma) = \dim H^{q(\lambda)+n-|\theta_{u,n}|}(Y, \mathcal{L}_\lambda) \tag{3.11}$$

where $q(\lambda)$ is given by (1.2), $2n = \dim_R G/K$ (i.e. $n = |\Delta_n^+|$); $q(\lambda) + n - |\theta_{u,n}| = q(\lambda,\theta)$ (see (3.4)) and (3.10) implies in fact that $Q_\lambda \subset \theta_{u,n}$. In general, apart from the hypothesis (3.10), (3.11) holds with $q(\lambda) + n - |\theta_{u,n}|$ replaced by $q(\lambda,\theta)$ if $S(\lambda, q(\lambda;\theta))$ in (3.6) reduces to a single element. The cohomology $H^*(Y,\mathcal{L}_\lambda)$ can survive only in dimensions sharply specified in Theorem 3.3.

PROOF. We shall exploit freely the notation of the preceding proof. As $\{\theta_i\}_{i=1}^t$ is a representative collection of parabolics $\supset \mathfrak{b}_\lambda$, $\theta_{u,n} \overset{(v)}{=} \theta_{u_{i_0},n}$ for some unique i_0, $1 \le i_0 \le t$; equivalently, for $w\theta \supset \mathfrak{b}_{\Lambda(\lambda)}$,

$$(w\theta)_{wu,n} = (w\theta_{i_0})_{wu_{i_0},n} = w\theta_{u_{i_0},n}.$$

Also

$$(\Lambda(\lambda) + \delta - \delta^{(\lambda)}, \Delta(wm)) = (w(\lambda + \delta - \delta^{(\lambda)}), w\Delta(m)) = (\lambda + \delta - \delta^{(\lambda)}, \Delta(m)) = 0,$$

by hypothesis, so $i_0 \in S(\Lambda(\lambda), q(\Lambda(\lambda); w\theta))$ by Proposition 2.7 (applied to $w\theta$), where

$$S(\Lambda(\lambda), q(\Lambda(\lambda); w\theta)) \equiv \underline{S}(\Lambda(\lambda), q(\Lambda(\lambda); w\theta))$$

(in the notation of the preceding proof)

$$\overset{(vi)}{\equiv} S(\lambda, q(\lambda;\theta)),$$

and where $\mu(\lambda, \theta_{i_0}) = \mu_{i_0} = w(\lambda + \delta - \delta^{(\Lambda)} + \langle \theta_{u_{i_0},n} \rangle) = w(\lambda + \delta - \delta^{(\lambda)} + \langle \theta_{u,n} \rangle)$, by (v). That is, $\mu(\lambda,\theta) = \mu_{i_0}$ by definition of $\mu(\lambda,\theta)$ and thus $\pi_{\mu(\lambda,\theta)} \overset{(viii)}{=} \pi_{\mu_{i_0}} = \pi_{\mu(\lambda,\theta_{i_0})}$ is unitarizable (as we have already observed that π_{μ_i} is unitarizable for any $i \in \underline{S}(\Lambda(\lambda), q), q \ge 0$) and $\mu(\lambda,\theta)$ is a $\overline{P}^{(\Lambda(\lambda))}$-highest weight; $\overline{P}^{(\Lambda(\lambda))} = \Delta_k^+ \cup -P_n^{(\lambda)}$. It should be noted that the unitarizability of $\pi_{\mu(\lambda,\theta)}$ also follows directly from [**4**] of course. Now if λ satisfies (3.10), then $\Lambda(\lambda)$ satisfies $(\Lambda(\lambda) + \delta - \delta^{(\lambda)}, \alpha) \neq 0$ for $\alpha \in Q_{\Lambda(\lambda)} \cup (w\theta)_{wu,n} = wQ_\lambda \cup w\theta_{u,n} = w(Q_\lambda \cup \theta_{u,n})$ so by Proposition 2.7 (again applied to $w\theta$), $Q_{\Lambda(\lambda)} \subset (w\theta)_{wu,n}$, $q(\Lambda(\lambda); w\theta) = |Q_{\Lambda(\lambda)}| + |\Delta_n^+| - |(w\theta)_{wu,n}|$, and $S(\Lambda(\lambda), q(\Lambda(\lambda); w\theta)) = \{i_0\}$. That is, $Q_\lambda \subset \theta_{u,n}$, $q(\lambda;\theta) = |Q_\lambda| + n - |\theta_{u,n}|$, and (by (vi)) $S(\lambda, q(\lambda;\theta)) \overset{(viii)}{=} \{i_0\}$. In particular,

$$\begin{aligned} q(\lambda,\theta) &\overset{\text{def.}}{=} |\{\alpha \in \Delta_k^+ : (\lambda + \delta, \alpha) < 0\}| + q(\lambda;\theta) \quad \text{(see (3.4))} \\ &= q(\lambda) + n - |\theta_{u,n}| \end{aligned}$$

(see (1.2) and (2.1)), and by (vii), (viii) equation (3.6) reduces to equation (3.11). If $S(\lambda, q(\lambda;\theta))$ reduces to a single element, even without assumption (3.10), then $S(\lambda, q(\lambda;\theta)) = \{i_0\}$ and again (vii) and (3.6) imply (3.11) provided $q(\lambda) + n - |\theta_{u,n}|$ in (3.11) is replaced, more generally, by $q(\lambda,\theta)$. Q.E.D.

REMARKS. We have seen that $\mu(\lambda,\theta)$ in Theorem 3.9 is given by $\mu(\lambda,\theta) = \Lambda(\lambda) + \delta - \delta^{(\Lambda(\lambda))} + w\langle\theta_{u,n}\rangle$ where $w \in W_k$ is the unique element such that $(w(\lambda+\delta), \Delta_k^+) > 0$ and $\Lambda(\lambda) = w(\lambda+\delta_k) - \delta_k \in \mathcal{F}_0'$ (see (2.2)). Note that $(w(\lambda+\delta), \Delta_k^+) = (w(\lambda+\delta_k), \Delta_k^+)$ since $(w\delta_n, \Delta_k^+) = (\delta_n, \Delta_k^+) = 0$. Also in Theorem 3.9, $\Delta_k^+ \cup -P_n^{(\lambda)} = w\overline{P}^{(\lambda)} = \overline{P}^{(\Lambda(\lambda))}$. Now suppose we choose $\theta = \mathfrak{b}_\lambda$ in Theorem 3.9. Then

$$\theta_{u,n} = P_n^{(\lambda)} \Rightarrow \mu(\lambda, \mathfrak{b}_\lambda) = \Lambda(\lambda) + \delta_n^{(\Lambda(\lambda))} + \delta_n$$

($P_n^{(\lambda)} = wP_n^{(\lambda)} = P_n^{(\Lambda(\lambda))}$). That is, $\mu(\lambda, \mathfrak{b}_\lambda)$ is the *lowest K-type* of the discrete series representation corresponding to the regular element $\Lambda(\lambda)+\delta$. The latter representation is unitarily equivalent to that corresponding to the regular element $\lambda+\delta$ since $\Lambda(\lambda)+\delta = w(\lambda+\delta)$ with $w \in W_k$. Thus $\pi_{\mu(\lambda,\mathfrak{b}_\lambda)}$ is the holomorphic discrete series representation of G corresponding to the regular element $\lambda+\delta$ and the holomorphic Weyl chamber $\Delta_k^+ \cup -P_n^{(\lambda)}$. Condition (3.10) then becomes condition (2.10), and (3.11) reduces to (1.3)(ii). Also in (3.4) (again for the choice $\theta = \mathfrak{b}_\lambda$)

$$2|\theta_{u,n} \cap Q_\lambda| - |\theta_{u,n}| + |Q_\lambda'| = 2|Q_\lambda| - n + |Q_\lambda'| = |Q_\lambda| \Rightarrow q(\lambda, \mathfrak{b}_\lambda) = q(\lambda)$$

in (1.2). That is, the vanishing result in Theorem 3.3 reduces to (1.3)(i).

A special case of interest is the case $Q_\lambda = \emptyset$; i.e. λ satisfies $(\lambda+\delta, \Delta_n^+) < 0$ or, equivalently $(\Lambda(\lambda)+\delta, \Delta_n^+) < 0$ since $w\Delta_n^+ = \Delta_n^+$. Then $P_n^{(\lambda)} = -\Delta_n^+ \Rightarrow$ every $\alpha \in P_n^{(\lambda)}$ is totally positive, and $q(\lambda;\theta)$ in (3.4) is $2\cdot 0 - |\theta_{u,n}| + n = n - |\theta_{u,n}|$, so that $q(\lambda,\theta) = |\{\alpha \in \Delta_k^+ : (\lambda+\delta,\alpha) < 0\}| + n - |\theta_{u,n}|$. (3.5) becomes for $q = q(\lambda;\theta) = n - |\theta_{u,n}|$

$$\begin{aligned} S(\lambda, q(\lambda;\theta)) = \{i : |\theta_{u_i,n}| = |\theta_{u,n}| \text{ and } \theta_{u_i,n} = \theta'_{u',n} \\ \text{for some parabolic subalgebra } \theta' = m' + u' \supset \mathfrak{b}_\lambda \\ \text{with } (\lambda+\delta-\delta^{(\lambda)}, \Delta(m')) = 0\} \end{aligned} \tag{3.12}$$

for a representative collection $\{\theta_i = m_i + u_i\}_{i=1}^t \supset \mathfrak{b}_\lambda$; cf. (2.3), (2.4). Also

$$P^{(\Lambda(\lambda))} = \Delta_k^+ \cup -\Delta_n^+ = \overline{\Delta}^+ \Rightarrow \delta - \delta^{(\Lambda(\lambda))} = 2\delta_n \Rightarrow \mu(\lambda,\theta) = \Lambda(\lambda) + 2\delta_n + w\langle\theta_{u,n}\rangle.$$

Thus Theorem 3.9 simplifies as follows.

THEOREM 3.13. *Let $\lambda \in \mathfrak{h}^*$ be an integral form such that $\lambda+\delta$ is regular and such that $(\lambda+\delta, \Delta_n^+) < 0$. Let $\theta = m + u \supset \mathfrak{b}_\lambda$ as in Theorem 3.9 with $(\lambda+\delta-\delta^{(\lambda)}, \Delta(m)) = 0$ or, equivalently, $(\Lambda(\lambda)+2\delta_n, w\Delta(m)) = 0$ in the above notation. Then the irreducible G-representation $\pi_{\mu(\lambda,\theta)}$ with Δ^+-highest weight $\mu(\lambda,\theta) = \Lambda(\lambda) + 2\delta_n + w\langle\theta_{u,n}\rangle$ is unitarizable and its multiplicity $m_{\pi_{\mu(\lambda,\theta)}}(\Gamma)$ in $L^2(\Gamma\backslash G)$ satisfies*

$$m_{\pi_{\mu(\lambda,\theta)}}(\Gamma) = \dim H^{q(\lambda)+n-|\theta_{u,n}|}(Y, \mathcal{L}_\lambda) \tag{3.14}$$

where $q(\lambda) = |\{\alpha \in \Delta_k^+ : (\lambda+\delta,\alpha) < 0\}| = |w(-\Delta_k^+) \cap \Delta_k^+|$, provided that either (i)′ *$(\lambda+\delta-\delta^{(\lambda)}, \alpha) \neq 0$ for $\alpha \in \theta_{u,n}$ (or, equivalently, $(\Lambda(\lambda)+2\delta_n, \beta) \neq 0$ for $\beta \in w\theta_{u,n}$) or* (ii)′ *the set $S(\lambda, q(\lambda;\theta))$ in* (3.12) *reduces to a single element. $H^q(Y, \mathcal{L}_\lambda) = 0$ unless $q = q(\lambda) + n - |\theta'_{u',n}|$ for some parabolic $\theta = m' + u' \supset \mathfrak{b}_\lambda$ such that $(\lambda+\delta-\delta^{(\lambda)}, \Delta(m')) = 0$.*

Note. In Theorem (3.13),

$$P^{(\lambda)} = w^{-1}P^{(\Lambda(\lambda))} = w^{-1}\Delta_k^+ \cup -w^{-1}\Delta_n^+ = w^{-1}\Delta_k^+ \cup -\Delta_n^+$$

(see equation ⓐ in the proof of Theorem 3.3) so that $\lambda + \delta - \delta^{(\lambda)} = \lambda + 2\delta_n + \delta_k - w^{-1}\delta_k$.

Finally we note that for $G = \mathrm{Sp}(m, R)$ or $\mathrm{SO}^*(2m)$ the set $S(\lambda, q(\lambda;\theta))$ in (3.12) reduces to a single element, for in these cases

$$\{|\theta_{u,n}| : \theta \supset \mathfrak{b}_\lambda\} = \left\{\frac{(m+1)m - (r+1)r}{2} : 0 \le r \le m\right\},$$

$$\left\{\frac{(m-1)m - (r-1)r}{2} : 1 \le r \le m\right\}, \quad \text{respectively;}$$

cf. Table 3.4 of [**7**]. Thus $|\theta_{u_r,n}| \neq |\theta_{u_j,n}|$ for $r \neq j$

COROLLARY 3.15. *For* $G = \mathrm{Sp}(m, R)$ $(m \ge 1)$ *or* $G = \mathrm{SO}^*(2m)$ $(m \ge 2)$ *condition* (ii)$'$ *in Theorem* 3.13 *is always satisfied. Hence* (3.14) *holds for* λ, θ *subject to the conditions of Theorem* 3.13 *prior to* (3.14).

In a future study we hope to extend the results of this paper to more general unitary representations having nontrivial relative $(\mathfrak{g}, K)$ cohomology.

REFERENCES

1. H. Fischer and F. Williams, *Borel-Le Potier diagrams—calculus of their cohomology bundles*, Tôhoku Math. J. **36** (1984), 233–251.
2. P. Griffiths and W. Schmid, *Locally homogeneous complex manifolds*, Acta Math. **123** (1969), 253–302.
3. R. Langlands, *Dimension of spaces of automorphic forms*, Proc. Sympos. in Pure Math., Vol. 9, Amer. Math. Soc., Providence, R.I., 1966, pp. 253–257.
4. R. Parthasarathy, *Criteria for the unitarizability of some highest weight modules*, Proc. Indian Acad. Sci. **89** (1980), 1–24.
5. W. Schmid, *On a conjecture of Langlands*, Ann. of Math. (2) **93** (1971), 1–42.
6. G. Warner, *Selberg's trace formula for nonuniform lattices: The R-rank one case*, Advances in Math., Suppl. Studies, 6, Academic Press, New York, 1979, pp. 1–142.
7. F. Williams, *Vanishing theorems for type* $(0,q)$ *cohomology of locally symmetric spaces* (I), Osaka J. Math. **18** (1981), 147–160.
8. ——, *An alternating sum formula for the multiplicities of unitary highest weight modules in* $L^2(\Gamma\backslash G)$, unpublished manuscript.
9. ——, *Discrete series multiplicities in* $L^2(\Gamma\backslash G)$ (II): *Proof of Langlands' conjecture*, Amer. J. Math. **107** (1985), 367–376.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MASSACHUSETTS 01003